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# FUNCTIONALS OF SUMMABLE FUNCTIONS.*

By William L. Hart.

**Introduction.** Let $H$ represent the class of all real valued functions $u(x)$, defined on an interval $a \leqq x \leqq b$ which, together with their squares $u^2(x)$, are summable in the Lebesgue sense on $(a, b)$. In the present paper we shall obtain certain results concerning functionals whose arguments are functions $u(x)$ of $H$. Functionals of this type have been considered previously† but the main results of the present paper are new and the point of view throughout is different from that of previous authors. In many instances below, the Riesz-Fischer theorem‡ concerning the Fourier constants of a summable function is appealed to in order to reduce questions concerning functionals defined in $H$ to related questions concerning functions of infinitely many variables defined in Hilbert space.

In Part I of the paper we shall consider functionals $F[u]$ which are continuous and possess differentials according to customary definitions. For the general linear functional, a representation by means of an infinite series is obtained. A mean value theorem is established for $F[u]$ and an infinite system of functional equations is solved in which the functionals involved are of type $F[u]$.

In Part II of the paper we shall discuss functionals $G[u, s]$ which, for every $u$ of $H$, are summable functions of $s$ defined on an interval $c \leqq s \leqq d$. In reference to $G[u, s]$, appropriate definitions are given for the terms *continuity* and *differential*. A representation for the general linear functional is obtained which involves an infinite series converging in the mean. The functions of infinitely many variables related to functionals $G[u, s]$ are shown to have partial *pseudo-derivatives* with respect to their arguments in the sense previously defined§ by the author. A mean value theorem for functionals $G[u, s]$ is obtained and a type of implicit functional equations is solved. A differential-functional equation is also considered where the derivative entering is a pseudo-derivative.

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* Presented to the American Mathematical Society at Chicago, Dec. 30, 1920.

† Lévy, Bulletin de la Société Mathématique de France, vol. XLVIII (1920), p. 13.

‡ Cf. Plancherel, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1910), p. 296.

§ Bulletin of the American Mathematical Society, vol. XXVII (1921), p. 202. Referred to in the future as Paper I.

All integrals below are taken in the sense of Lebesgue. A function $f(x)$ will be termed *integrable* if both $f$ and $f^2$ are summable in the Lebesgue sense. All functions and variables will be supposed real valued. The phrase *almost everywhere* will mean *with the exception at most of a set of points of measure zero*. In the discussion below two functions $h_1(x)$ and $h_2(x)$, defined on a set $E$, well be called the same if they are equal almost everywhere on $E$. In all equations, as for example $h_1(x) = h_2(x)$, it will be understood that the equality may fail to hold on some sub-set of $E$ of measure zero.

If $u(x)$ is a function of $H$ we shall denote the positive square root of $\int_{(ab)} u^2(x)\,dx$ by $Mu$ and shall call it the modulus of $u(x)$. We shall have occasion many times to refer to a function $u(x)$ of $H$ as the *point* $u(x)$, and we may think of the modulus of $u(x)$ as representing the distance of $u(x)$ from the origin, $u = 0$, in $H$.

Let $I$ represent a sequence of integrable functions $[p_i(x);\ i = 1, 2, \cdots]$, which are unitary and orthogonal on the interval $(a, b)$ and which, moreover, form a complete set for the class $H$. That is, there does not exist in $H$ any function $u(x) \neq 0$ for which $z_i = 0$ $(i = 1, 2, \cdots)$, where

$$z_i = \int_{(ab)} u(x)\,p_i(x)\,dx. \tag{1}$$

We shall call the numbers $z_i$ the *Fourier coefficients* of $u(x)$ and shall term $\zeta = (z_1, z_2, \cdots)$ the *Fourier vector* corresponding to $u(x)$. It is well known that $(Mu)^2 = \sum_{i=1}^{\infty} z_i^2$ and, accordingly, we shall call the positive square root of this infinite sum the modulus of $\zeta$ and shall represent it by $M\zeta$. Since $M\zeta$ exists, the vector $\zeta$ may be considered as representing a point in real Hilbert space of infinitely many dimensions. Moreover, as a consequence of the Riesz-Fischer theorem, it is known that a one to one correspondence exists between points $u(x)$ in $H$ and points $\zeta$ in Hilbert space. If $\zeta$ is the point corresponding to a function $u(x)$ we shall denote this fact by the notation $\zeta \equiv u(x)$.

It will be useful later to recall that, if $u_n = z_1 p_1 + \cdots + z_n p_n$, then

$$\lim_{n=\infty} M^2(u_n - u) = \lim_{n=\infty} \sum_{i=n+1}^{\infty} z_i^2 = 0. \qquad (\zeta \equiv u(x)) \tag{2}$$

Moreover, if $(u_n(x);\ n = 1, 2, \cdots)$ is a sequence of $H$ satisfying

$$\lim_{n=\infty} M(u_n - u) = 0, \tag{3}$$

it is easily verified that

$$\lim_{n=\infty} M(\zeta_n - \zeta) = 0. \qquad (\zeta_n \equiv u_n;\ \zeta \equiv u) \tag{4}$$

When a sequence $(u_n)$ satisfies (3) it is customary to say that $u$ is the *limit in the mean** of the sequence $(u_n)$. We shall use the notation

$$\operatorname*{l.\,m.}_{n=\infty} u_n = u, \tag{5}$$

to denote such convergence. Similarly, we shall abbreviate the convergence in (4) by the notation $\operatorname*{l.\,m.}_{n=\infty} \zeta_n = \zeta$. If the partial sums $S_n(x)$ of a series $\sum_{i=1}^{\infty} u_i(x)$ of functions of $H$ satisfy the equation $\operatorname*{l.\,m.}_{n=\infty} S_n(x) = S(x)$, we shall say that the series converges in the mean to $S(x)$ and we shall indicate this fact by the notation.

$$\sum_{i=1}^{\infty}{}^{(m)} u_i(x) = S(x). \tag{6}$$

## PART I.

## Functionals without a Parameter.

**1. Definitions and theorems on linear functionals.** Let $F[u]$ be a functional operation which, for every $u$ in $H$, yields a real number. Then we shall say that $F[u]$ is continuous† at $u$ if, whenever equation (5) is satisfied, it follows that $\lim_{n=\infty} F[u_n] = F[u]$. It is seen immediately that, if $u_1(x) = u_2(x)$ almost everywhere, then $F[u_1] = F[u_2]$. In the future all functionals considered will be supposed continuous.

A functional $L[u]$ is linear if, for every pair of points $u_1$ and $u_2$ in $H$, $L[u_1 + u_2] = L[u_1] + L[u_2]$. It is easily verified that, for every real constant $c$ and for every $u$ of $H$, $L[cu] = cL[u]$.

*Definition 1.*‡ *Suppose that, corresponding to a point $u_0$ in $H$, there exists a linear functional $L[u_0, v]$, defined for all functions $v(x)$ in $H$ and satisfying the equation*

$$F[u] = F[u_0] + L[u_0, v] + (Mv)\, e(u_0, v), \tag{7}$$

*where $u = u_0 + v$ and where $e(u_0, v)$, approaches zero with $Mv$. Then, we shall call $L[u_0, v]$ the differential of $F$ at the point $u_0$.*

It is obvious that the differential is unique, if it exists, because, as a consequence of (7),

$$L[u_0, v] = \lim_{d=0} \frac{F[u_0 + dv] - F[u_0]}{d}.$$

* Cf. Plancherel, loc. cit., p. 292.

† Cf. Lévy, loc. cit., p. 21.

‡ Cf. Lévy, loc. cit., p. 21.

Let $f(\zeta)$ be a function whose value, for every $\zeta$ in Hilbert space, is defined by the equation $f(\zeta) = F[u]$ where $u \equiv \zeta$. We shall say that $f(\zeta)$ *corresponds to* $F[u]$ *and shall* denote this fact by the notation $f(\zeta) \equiv F[u]$. In view of equations (3) and (4) it is seen that $f(\zeta)$ is continuous in the weak* sense at every point $\zeta$ in Hilbert space.

THEOREM 1. *If* $L[u]$ *is a linear functional, then*

$$L[u] = \sum_{i=1}^{\infty} z_i L[p_i]. \qquad (\zeta \equiv u) \tag{8}$$

As a consequence of the linearity and the continuity of $L[u]$, it follows that

$$L[u] = \lim_{n=\infty} L[z_1 p_1 + \cdots + z_n p_n] = \lim_{n=\infty} \sum_{i=1}^{n} z_i L[p_i],$$

which establishes the theorem.

Since (8) must converge for all points $\zeta$ in Hilbert space, we may state as an immediate corollary that the series

$$\sum_{i=1}^{\infty} (L[p_i])^2 \tag{9}$$

converges.† Moreover, it is seen that the function $l(\zeta)$, corresponding to $L[u]$, is given by the right member of (8) and hence is a linear form in infinitely many variables.

*Corollary 1.‡ If* $L[u]$ *is linear, there exists a constant* $K > 0$ *such that, for all* $u$ *of* $H$, $|L[u]| \leqq KMu$.

If we apply the Schwarz inequality for sums in (8) we obtain

$$|L[u]| \leqq (Mu) \left[ \sum_{i=1}^{\infty} (L[p_i])^2 \right]^{\frac{1}{2}},$$

from which the corollary follows with an obvious definition for $K$.

If we insert expressions (1) in (8), we immediately obtain the equation

$$L[u] = \lim_{n=\infty} \int_{(ab)} K_n(x)\, u(x)\, dx \qquad (K_n(x) = p_1(x) L[p_1] + \cdots + p_n(x) L[p_n])$$

It should be noted that the functions $(K_n)$ do not depend on $u$.

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* Cf. Hart, Transactions of the American Mathematical Society, vol. XXIII (1922), p. 32. This paper will be referred to in the future as Paper II.

† Cf. Riesz, *Équationes Linéaires*, p. 47.

‡ Cf. Lévy, loc. cit., p. 20.

THEOREM 2. *Suppose that $F[u]$ possesses a differential $L[u, v]$ at the point $u$ in $H$. Then, if $f(\zeta) \equiv F[u]$, it follows that, at the point $\zeta$ in Hilbert space corresponding to $u$, there exist partial derivatives*

$$\frac{\partial f(\zeta)}{\partial z_i} = L[u, p_i]. \qquad (i = 1, 2, \cdots) \tag{10}$$

*Moreover, the following series converges:*

$$\sum_{i=1}^{\infty}\left(\frac{\partial f(\zeta)}{\partial z_i}\right)^2. \tag{11}$$

In order to establish (10) for the case $i = 1$, for example, we start from the equation

$$\begin{aligned} f(z_1 + d, z_2, z_3, \cdots) - f(\zeta) &= F[u + dp_1] - F[u], \\ &= L[u, dp_1] + |d|(Mp_1)e(u, dp_1), \\ &= dL[u, p_1] + |d|(Mp_1)e(u, dp_1). \end{aligned}$$

Hence, since $\lim_{d=0} M(dp_1) = 0$, it follows that (10) is true for $i = 1$. A similar proof establishes the result for all values of $i$. The convergence of (11) is a direct consequence of (9).

The conclusions of Theorem (1), applied to the differential $L[u; v]$, permit us to state a corresponding result for $F[u]$.

THEOREM 3. *Under the hypotheses of Theorem 2, it follows that*

$$F[u'] = F[u] + \sum_{i=1}^{\infty}(z_i' - z_i)\, L[u, p_i] + (Mv)\, e(u, v),$$

*where $v = u' - u$ and where $u' \equiv \zeta' = (z_1', z_2', \cdots)$.*

**2. A mean value theorem and the solution of functional equations.** With the aid of the results of the previous section we may obtain several theorems in regard to functionals by the use of theorems previously proved by the author* for functions defined in Hilbert space.

Let $T[u, v]$ be a functional defined for all $(u, v)$ in $H$. We shall say that $T$ is continuous simultaneously in its arguments if the equations

$$\underset{n=\infty}{\text{l.m.}}\, u_n = u, \qquad \underset{n=\infty}{\text{l.m.}}\, v_n = v,$$

where $(u_n, v_n, u, v)$ are in $H$, imply that $\lim_{n=\infty} T[u_n, v_n] = T[u, v]$.

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* Cf. Paper II.

THEOREM 4. *Let $F[u]$ possess a differential $L[u,v]$ for all $u$ in $H$ and suppose that $L$ is continuous simultaneously in its arguments for all $(u, v)$ in $H$. Assume, moreover, that a constant $K > 0$ exists such that, for all $(u, v)$,*

$$|L[u, v]| \leqq KMv. \tag{13}$$

*Then it follows that, if $(u', u)$ are in $H$,*

$$F[u'] - F[u] = \int_0^1 L[u + tv, v]\, dt \qquad (v = u' - u) \tag{14}$$

$$= \sum_{i=1}^{\infty} (z_i' - z_i) \int_0^1 L[u + tv, p_i]\, dt. \qquad (u \equiv \zeta;\, u' \equiv \zeta') \tag{15}$$

Since $L$ is linear in $v$, we know that, for every $u$, a constant $K(u)$ exists satisfying (13). The force of the assumption (13) lies in the supposition that $K(u) \leqq K$ for all $u$. By virtue of (13), since $L$ is a linear form in the quantities $(z_i' - z_i)$, a well known theorem* permits us to state that, for every $u$,

$$\sum_{i=1}^{\infty} (L[u, p_i])^2 \leqq K^2. \tag{16}$$

Suppose, now, that two points $(u', u)$ in $H$ are assigned and let us establish (14). It is seen that, for a given pair $(u, v)$, the function of $r$ defined by $S(r) = F[u + rv]$ is continuous and has a continuous derivative $L[u + rv, v]$ for $0 \leqq r \leqq 1$. Consequently, the mean value theorem in integral form, applied to the difference $S(1) - S(0)$, establishes (14).

In considering (15) we first note that, because of (16), the series

$$\sum_{i=1}^{\infty} (z_i' - z_i)\, L[u + tv, p_i] \tag{17}$$

converges uniformly for $0 \leqq t \leqq 1$. Since the integrand in (14) is equal to (17), it follows that the right member of (15) exists and is equal to that of (14).

The theorem could also have been proved by an application of the mean value theorem for functions in Hilbert space.†

Let us consider the infinite system of equations

$$F_i[u, t] = 0, \qquad (i = 1, 2, \cdots) \tag{18}$$

where, for every $t$ on an interval $c \leqq t \leqq d$, $F_i[u, t]$ is a functional of the type $F$ treated previously. Suppose that, for every $t$, $F_i[u, t]$ possesses

* Cf. Hellinger and Toeplitz, Mathematische Annalen, vol. LXIX (1910), p. 295.

† Paper II, Theorem II.

a differential $L_i[u, v; t]$ at every point $u$ in $H$, and assume that both $F_i$ and $L_i$ are continuous in their arguments simultaneously. That is, suppose that the equations

$$\underset{n=\infty}{\text{l.m.}}\, u_n = u, \qquad \underset{n=\infty}{\text{l.m.}}\, v_n = v, \qquad \lim_{n=\infty} (t_n - t) = 0,$$

where $(u_n, v_n, u, v)$ are in $H$ and where $(t_n, t)$ are on $(c, d)$, imply that

$$\lim_{n=\infty} F_i[u_n, t_n] = F_i[u, t], \qquad \lim_{n=\infty} L_i[u_n, v_n; t_n] = L_i[u, v; t].$$

Under these assumptions it is seen that, if $f_i(\zeta, t) \equiv F_i[u, t]$ and $l_i(\zeta, v; t) \equiv L_i[u, v; t]$, then

$$\frac{\partial f_i(\zeta, t)}{\partial z_j} = L_i[u, p_j; t] = l_i[\zeta, p_j; t]. \qquad (i, j = 1, 2, \cdots; \zeta \equiv u) \tag{19}$$

Therefore, the derivatives (19) are continuous in their arguments simultaneously, in the weak sense with respect to $\zeta$ and in the ordinary sense with respect to $t$.

The previous theorems of this paper together with the theorem* on implicit functions for Hilbert space applied to the system

$$f_i(\zeta, t) = 0, \qquad (i = 1, 2, \cdots)$$

permit us to state the following theorem without further proof.

THEOREM 5. *Assume that the series*

$$\sum_{i=1}^{\infty} F_i^2[u, t], \sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} L_i^2[u, p_j; t], \sum_{i=1}^{\infty} (L_i[u, p_i; t] - 1)^2, \tag{20}$$

*converge uniformly for all $u$ in $H$ and for $c \leqq t \leqq d$. Suppose, moreover, that the infinite determinant*

$$\Delta = |L_i[u_0, p_j; t]|_{i,j=1,2,\ldots} \tag{21}$$

*is not zero. Then, if $(u = u_0, t = t_0)$ is a solution of (18) we can select a number $r > 0$ so small that, for $|t - t_0| \leqq r$, there exists uniquely a function $u(x, t)$ satisfying (18). For every $t$ on $|t - t_0| \leqq r$ the function $u(x, t)$ is in $H$. The coordinates $z_i(t)$ of the point $\zeta(t) \equiv u(x, t)$ are continuous functions of $t$ and*

$$\lim_{t=t_0} M\zeta(t) = \lim_{t=t_0} Mu(x, t) = Mu_0. \tag{22}$$

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* Cf. Paper II, Theorem V.

## PART II.

## Functionals Containing a Parameter.

1. **Definitions and theorems on linear functionals.** Let $W$ represent the class of all functions $w(x)$ which are integrable on an interval $E=(c \leqq s \leqq d)$. Suppose that $G[u, s]$ is a functional operation which, for every $u$ of $H$, yields a function of $s$ belonging to the class $W$.

*Definition* 1.* *The functional $G[u, s]$ is continuous at $u$ if, whenever* $\underset{n=\infty}{\text{l. m.}}\, u_n = u$, *it follows that*

$$\underset{n=\infty}{\text{l. m.}}\, G[u_n, s] = G[u, s]. \tag{23}$$

*The functional is linear if $G[u_1, s] + G[u_2, s] = G[u_1 + u_2, s]$ for all $(u_1, u_2)$ in $H$.*

It should be recalled that the statement (23) implies that

$$\lim_{n=\infty} \int_E (G[u_n, s] - G[u, s])^2\, ds = 0.$$

All functionals considered below will be supposed continuous. It is seen that, if $u_1(x) = u_2(x)$ almost everywhere, then $G[u_1, s] = G[u_2, s]$. Moreover, if $L[u, s]$ is a linear functional, it is easily established that $L[cu, s] = cL[u, s]$ for every constant $c$ and for $u$ in $H$. It is known† that, for every linear functional $L[u, s]$, there exists a constant $K > 0$ such that

$$ML[u, s] = \sqrt{\int_E L^2[u, s]\, ds} \leqq KMu.$$

THEOREM 1. *If $L[u, s]$ is a linear functional, then, for every $u$ in $H$,*

$$L[u, s] = \sum_{i=1}^{\infty}(m)\, z_i\, L[p_i, s], \qquad (\zeta \equiv u) \tag{24}$$

$$= \underset{n=\infty}{\text{l. m.}} \int_{(ab)} V_n(x, s)\, u(x)\, dx, \tag{25}$$

where $V_n(x, s) = p_1(x)\, L[p_1, s] + \cdots + p_n(x)\, L[p, s]$.

The proof of (24) would make use of the continuity of $L$ and would be similar in method to the proof. of Theorem (1), Part I, with "limit in the mean" replacing "limit" as met in Part I. Equation (25) is obtained from (24) by inserting the expressions (1) for the $z_i$. It should be noted that $V_n(x, s)$ is independent of $u(x)$. Moreover, from a known property of convergence in

* Cf. Lévy, loc. cit., p. 20.

† Cf. Lévy, loc. cit., p. 20.

the mean,* it follows that a sub-sequence ($V_{h_n}$; $n = 1, 2, \cdots$) can be selected from the sequence ($V_n$) so that

$$\lim_{n=\infty} \int_{(ab)} V_{h_n}(x,s)\,u(x)\,dx = \lim_{n=\infty} \sum_{i=1}^{h_n} z_i L[p_i, s] = L[u,s],$$

almost everywhere on $E$.

Let $L[u_0; v, s]$, where $u_0$ is a point in $H$, be a linear functional defined for $v$ in $H$ and for $s$ on $E$.

*Definition 2.*† *Suppose that the functional $G$ satisfies the equation*

$$G[u,s] - G[u_0,s] = L[u_0; v,s] + (Mv)\,e(u_0, v, s), \qquad (v = u - u_0) \tag{26}$$

*where $Me(u_0, v, s)$ approaches zero with $Mv$. Then, $L[u_0; v, s]$ is defined as the differential of $G[u,s]$ at the point $u_0$.*

The uniqueness of the differential $L$, if it exists, is easily established by use of the notion of a pseudo-derivative as previously defined by the author.‡ Let $R(r,s)$ be integrable on $E$ for every $r$ on an interval $T$. Then, the pseudo-derivative of $R$ with respect to $r$ at the point $r = r_0$ is the integrable function $R_r(r_0, s)$ which satisfies the equation

$$\underset{h=0}{\mathrm{l.\,m.}} \frac{\Delta R}{h} = R_r(r_0, s), \qquad [\Delta R = R(r_0 + h, s) - R(r_0, s)] \tag{27}$$

provided that the limit exists. It should be recalled that (27) implies that

$$\lim_{h=0} \int_E \left(R_r(r_0, s) - \frac{\Delta R}{h}\right)^2 ds = 0.$$

Moreover, when the pseudo-derivative at a point $r_0$ exists, it is unique up to its values at a set of points $s$ of measure zero.

THEOREM 2. *Suppose that $G$ possesses a differential $L$ at all points in $H$. Let $R(r,s) = G[u + rv, s]$, where $0 \leqq r \leqq 1$. Then, for every value of $r$ there exists a pseudo-derivative $R_r(r,s) = L[u + rv; v, s]$.*

In order to establish the theorem let us first note that, if $\lim_{n=\infty} r_n = r$, then $\underset{n=\infty}{\mathrm{l.\,m.}}\, u + r_n v = u + rv$. Consider the expression

$$\begin{aligned} \frac{\Delta R}{h} &= \frac{L[u + rv; vh, s]}{h} + \frac{M(hv)}{h} e(u + rv, hv, s), \quad [\Delta R = R(r+h, s) - R(r,s)] \\ &= L[u + rv; v, s] + (Mv)\,e(u + rv, hv, s). \end{aligned}$$

* Cf. Plancherel, loc. cit., p. 294.

† Cf. Lévy, loc. cit., p. 21.

‡ Paper I. Definition (3).

Consequently, as a result of the properties of $e(u, v, s)$, it follows that

$$\lim_{h=0} \int_E \left(\frac{\Delta R}{h} - L[u + rv; v, s]\right)^2 ds = 0,$$

which establishes the theorem.

When we place $r = 0$ in Theorem 2 we obtain $R_r(0, s) = L[u; v, s]$. Hence, since the pseudo-derivative is unique, it follows that the differential $L[u; v, s]$ is unique, if it exists.

2. **Expressions for $G[u, s]$ in terms of its differential.** The mean value theorem* for a function possessing a pseudo-derivative, applied to $R(r, s)$ enables us to obtain a mean value theorem for $G$ without further proof.

THEOREM 3. *Let $G[u, s]$ satisfy all conditions of Theorem 2 and suppose that*

$$\lim_{r_1 = r} \int_E (L[u + r_1 v; v, s] - L[u + rv; v, s])^2 ds = 0,$$

*for all values of $r$ on $(0,1)$. Assume that, for every pair $(u, v)$, $L[u + rv; v, s]$ is integrable with respect to $(r, s)$ in the rectangle $(0 \leqq r \leqq 1; c \leqq s \leqq d)$. Then it follows that, for every $(u_1, u)$ in $H$,*

$$G[u_1, s] - G[u, s] = \int_{(0,1)} L[u + rv; v, s] dr. \qquad (v = u_1 - u) \tag{28}$$

Since $L[u; v, s]$ is linear in $v$, there exists, for every $u$, a constant $K(u)$ statisfying the equation

$$M(L[u; v, s]) \leqq K(u) M v.$$

*Corollary 1. Assume that the hypotheses of Theorem 3 are satisfied and that a constant $K > 0$ exists such that $K(u) \leqq K$ for all $u$ of $H$. Then, if $u_1 = u + v$,*

$$G[u_1, s] - G[u, s] = \sum_{i=1}^{\infty}(m)(z_{i1} - z_i) \int_{(0,1)} L[u + rv; p_i, s]\, dr. \qquad [u_1 \equiv (z_{11}, z_{21}, \cdots)]$$

Since $L$ is linear, we know that

$$L[u + rv; v_n, s] = \sum_{i=1}^{n} z'_i L[u + rv; p_i, s], \qquad (z'_i = z_{i1} - z_i)$$

where $v_n = z'_1 p_1 + \cdots + z'_n p_n$. Moreover,

$$L[u + rv; v, s] - L[u + rv; v_n, s] = L[u + rv; v - v_n, s]. \tag{29}$$

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* Paper I, Theorem IX.

Consequently, for a fixed value of $r$, the modulus of the left member of (29) is at most

$$KM(v - v_n), \tag{30}$$

and hence approaches zero for $n = \infty$, uniformly for $0 \leqq r \leqq 1$.

In view of (28) it follows that Corollary (1) will be established if we show that

$$\lim_{n=\infty} \int_E \Big[ \int_{(0,1)} (L[u + rv; v, s] - L[u + rv; v_n, s])\, dr \Big]^2 ds = 0. \tag{31}$$

By an application of the Schwarz inequality to the inner integral in (31), it is seen that, for a given value of $n$, the integral in (31) is at most

$$\int_E \Big[ \int_{(0,1)} L^2[u + rv; v - v_n, s]\, dr \Big]\, ds \tag{32}$$
$$= \int_{(0,1)} dr \int_E L^2[u + rv; v - v_n, s]\, ds \leqq K^2(d - c)\, M^2(v - v_n).$$

The interchange of integrations made in obtaining (32) was permissible* under our present hypotheses. The inequality in (32) is a result of (30). From (32) it is evident that (31) is true.

**3. The Fourier coefficients of** $G[u, s]$. Let $(y_i(s); i = 1, 2, \cdots)$ be a system of functions, unitary and orthogonal on $E$ and complete for the class $W$. For a given function $u$ of $H$, $G[u, s]$ will possess Fourier coefficients $(F_i[u]; i = 1, 2, \cdots)$, where

$$F_i[u] = \int_E G[u, s]\, y_i(s)\, ds, \tag{33}$$

which satisfy the equation

$$\int_E G^2[u, s]\, ds = \sum_{i=1}^{\infty} F_i^2[u].$$

It is obvious that, if $G[u, s]$ is a linear functional, then $F_i$ is linear in the sense of Part I. Further properties of the $(F_i)$ are considered in the next theorem.

THEOREM 4. *The functionals $(F_i)$ are continuous in the sense of Part I. Furthermore, if $G[u, s]$ possesses a differential at $u_0$ in $H$, the functionals $(F_i)$ possess differentials at $u_0$ in the sense of Part I.*

As a consequence of (33), it is seen that

$$\int_E (G[u_n, s] - G[u, s])^2\, ds = \sum_{i=1}^{\infty} (F_i[u_n] - F_i[u])^2. \tag{34}$$

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* Cf. de la Vallée Poussin, *Intégrales de Lebesgue*, p. 53.

If a sequence $(u_n)$ satisfies $\underset{n=\infty}{\text{l. m.}}\, u_n = u$, the whole expression (34), and, therefore, every term in the infinite series, approaches zero for $n = \infty$. This establishes the continuity of the $(F_i)$.

If $G$ possesses a differential at $u_0$, it is seen from equation (26) that

$$F_i[u] - F_i[u_0] = L_i[u_0; v] + Mv \int_E e(u_0, v, s)\, y_i(s)\, ds, \tag{35}$$

where $L_i$ is the $i$th Fourier coefficient of $L$. The equation (35) shows that $L_i$ is the differential of $F_i$ provided that the integral entering in (35) approaches zero with $Mv$. This fact is easily established by use of the properties of $e(u_0, v, s)$, because

$$\left| \int_E e(u_0, v, s)\, y_i(s)\, ds \right|^2 \leqq \left( \int_E y_i^2(s)\, ds \right) \int_E e^2(u_0, v, s)\, ds.$$

This completes the proof of Theorem 4.

4. **The functions in Hilbert space corresponding to** $G[u, s]$. Let us define a function $g(\zeta, s)$, in which $\zeta$ is in Hilbert space, by the equation $g(\zeta, s) = G[u, s]$, where $u \equiv \zeta$. We shall say that $g(\zeta, s)$ corresponds to $G[u, s]$ and shall denote this fact by the notation $g(\zeta, s) \equiv G[u, s]$. With the aid of equations (3) and (4) it is verified that, if $\underset{n=\infty}{\text{l.m.}}\, \zeta_n = \zeta$, then $\underset{n=\infty}{\text{l.m.}}\, g(\zeta_n, s) = g(\zeta, s)$. If $L[u, s]$ is linear, its corresponding function $l[\zeta, s]$ is given by the right member of (24) and hence is a linear form in the variables $(z_1, z_2, \cdots)$ converging in the mean.

THEOREM 5. *At the point $u$ in $H$ suppose that $G[u, s]$ possesses the differential $L[u; v, s]$, and let $g(\zeta, s) \equiv G[u, s]$. Then, at the point $\zeta \equiv u$, there exist partial pseudo-derivatives $g_{z_i}(\zeta, s)$ given by the equations*

$$g_{z_i}(\zeta, s) = l(\zeta; p_i, s) \qquad (l(\zeta; p_i, s) \equiv L[u; p_i, s];\ i = 1, 2, \cdots).$$

For simplicity consider only the case $i = 1$. Let $u_1 = u + hp_1$. With the aid of (26) we obtain

$$\frac{\Delta G}{h} = \frac{g(z_1 + h, z_2, z_3, \cdots, s) - g(\zeta, s)}{h} = \frac{G[u_1, s] - G[u, s]}{h}$$
$$= L[u; p_1, s] + e(u, hp_1, s),$$

where use has been made of the fact that $Mp_1 = 1$. As a consequence of the properties of $e(u, v, s)$ it is seen that

$$\lim_{h=0} \int_E \left( \frac{\Delta g}{h} - L[u; p_1, s] \right)^2 ds = 0,$$

which establishes the theorem.

**5. An implicit functional equation.** Consider the functional equation

$$G[u; s, t] = 0, \tag{36}$$

where, for every $t$ on an interval $|t - t_0| \leq k$, the functional $G[u; s, t]$ is of the type considered previously in this part of the paper. Suppose that $u = u_0$, $t = t_0$ satisfies (36). Then, we shall seek to determine a function $u(x, t)$ which, for every $t$, is a function of $H$ and which has the property that $G[u(*, t); s, t]$, when $t$ is fixed, is zero almost everywhere on $E$. The argument $x$ in $u(x, t)$ was replaced here by "$*$" in order to emphasize the character of the dependence of $G$ on $u$. The Riesz-Fischer theorem shows that the problem proposed is equivalent to the problem solved in Theorem 5, Part I, for the system

$$F_i[u, t] = 0. \qquad \left(i = 1, 2, \cdots; \; F_i[u, t] = \int_E G[u; s, t]\, y_i(s)\, ds\right) \tag{37}$$

In regard to (36) we shall make the following assumptions:

(a) For every $t$, $G[u; s, t]$ possesses a differential $L[u; v, s, t]$ at all points $u$ in $H$.

(b) If $\underset{n=\infty}{\text{l. m.}}\, u_n = u$, $\underset{n=\infty}{\text{l. m.}}\, v_n = v$, and $\lim_{n=\infty} t_n = t$, then

$$\underset{n=\infty}{\text{l. m.}}\, G[u_n; s, t_n] = G[u; s, t],$$
$$\underset{n=\infty}{\text{l. m.}}\, L[u_n; v_n, s, t_n] = L[u; v, s, t].$$

(c) The series (20), formed for the system (37), converge uniformly for $u$ in $H$ and for $|t - t_0| \leq k$.

From (a), with the assistance of Theorem 4, it it seen that $F_i$ in (37) possesses the differential

$$L_i[u; v, t] = \int_E L[u; v, s, t]\, y_i(s)\, ds.$$

With the aid of (b) an application of the Schwarz inequality shows that

$$\begin{aligned}(F_i[u_n, t_n] - F_i[u, t])^2 &= \left[\int_E (G[u_n; s, t_n] - G[u; s, t])\, y_i(s)\, ds\right]^2, \\ &\leq \int_E (G[u_n; s, t_n] - G[u; s, t])^2 ds,\end{aligned}$$

which approaches zero for $n = \infty$. Hence, $F_i$ has the continuity property postulated for system (18) of Part I. It may be shown in a similar fashion that the differential $L_i$ of the present section has the continuity assumed for the $L_i$ of that system.

In (c) it should be remarked that the first series of (20) always converges for a system (37) since the $F_i$ are the Fourier coefficients of $G$. The force of the condition (c) lies in the assumption of uniformity for this convergence. The uniform convergence of the third series of (20) for system (37) would be insured if we should postulate uniform convergence for the series

$$\sum_{i=1}^{\infty} \int_E (L[u, p_i, s, t] - y_i(s))^2 ds.$$

The consequences we have just derived from the assumptions (a), (b), and (c) permit us to state the next theorem without further proof.

THEOREM 6. *Suppose that* (36) *satisfies* (a), (b), *and* (c), *and that the infinite determinant* (21), *formed for the system* (37) *is different from zero. Then, the hypotheses of Theorem 5, Part I, are satisfied by* (37), *and the concluding statement of that theorem in regard to the existence of a solution* $u(x,t)$ *applies to* (36).

6. **A pseudo-differential equation.** For every value of $t$ on $|t-t_0| \leqq k$ suppose that the function $u(x,t)$ is a function of $H$ on the interval $(a,b)$. Consider the equation

$$u_t(x,t) = G[u(*,t);\, t,x], \tag{38}$$

where for every $t$, $G[u;t,x]$ is a functional operation of the type defined in § 1, with the interval $E$ replaced by the interval $(a,b)$. In (38) $u_t$ represents the pseudo-derivative of $u$ with respect to $t$, and it should be remembered that, by the definition of the pseudo-derivative, $u_t(x,t)$ is required to be a function of $H$ for every value of $t$. The "$*$" is used in (38) to emphasize the fact that $G$ is a functional of $u$.

Let $P$ represent the class of all functions $u(x,t)$ which possess pseudo-derivatives $u_t(x,t)$ for some values of $t$. Then, we wish to state hypotheses under which (38) will have a solution $u(x,t)$ belonging to $P$ and satisfying a given initial condition $u(x,0) = u_0(x)$, where $u_0(x)$ is in $H$.

Before discussing (38) let us recall certain facts about pseudo-derivatives previously established by the author*. Let $\zeta(t) = [z_1(t), z_2(t), \cdots]$ and $\eta(t) = [k_1(t), k_2(t), \cdots]$ belong to Hilbert space for every $t$, and let $u$ and $w$ be functions of $(x,t)$ satisfying the relations $\zeta(t) \equiv u(x,t)$, $\eta(t) \equiv w(x,t)$. If $w(x,t) = u_t(x,t)$ then $dz_i/dt = k_i$ and, moreover,

$$\lim_{h=0} \sum_{i=1}^{\infty} \left( \frac{z_i(t+h) - z_i(t)}{h} - k_i(t) \right)^2 = 0, \tag{39}$$

* Paper I, Theorems VII and VIII.

for every value of $t$. Conversely, if $\zeta(t)$ and $\eta(t)$ satisfy (39), it follows that the pseudo-derivative $u_t$ exists and is defined by the equation $u_t(x,t) = w(x,t)$. In view of these statements it is seen that, if $u(x,t)$ is in the class $P$ and if $\zeta(t) \equiv u(x,t)$, then $d\zeta(t)/dt \equiv u_t(x,t)$, where

$$\frac{d\zeta(t)}{dt} = \left(\frac{dz_1(t)}{dt}, \frac{dz_2(t)}{dt}, \cdots\right).$$

For every $t$ and $u$ of $H$ define the functional $G_i[u,t]$ by the equation

$$\begin{aligned} G_i[u,t] &= \int_{(ab)} G[u;\, t, x]\, p_i(x)\, dx, \\ &= g_i(\zeta, t). \qquad (u \equiv \zeta;\; g_i(\zeta, t) \equiv G_i[u,t]) \end{aligned}$$

It is seen that, if (38) is satisfied by a function $u(x,t)$ of $P$, the point $\zeta(t) \equiv u(x,t)$ satisfies the system of equations

$$\frac{dz_i}{dt} = g_i(\zeta, t). \qquad (i = 1, 2, \cdots) \tag{40}$$

Conversely, if $\zeta = \zeta(t)$ is a solution of (40) whose coordinates $z_i(t)$ satisfy (39) with $k_i$ replaced by $dz_i/dt$, it follows that the function $u(x,t)$ corresponding to $\zeta(t)$ is in the class $P$ and satisfies (38). A system (40) whose solution has the desired property (39) has been treated previously by the author.* Sufficient conditions will now be imposed on (38) so that the system (40) corresponding to it can be treated by the theorem referred to.

Assume in the future that $G[u;\, t, x]$ satisfies the following conditions:

(a) If $\underset{n=\infty}{\text{l. m.}}\, u_n(x) = u(x)$ and $\lim\limits_{n=\infty} t_n = t$, then

$$\underset{n=\infty}{\text{l. m.}}\, G[u_n;\, t_n, x] = G[u;\, t, x].$$

(b) There exist positive constants $A$ and $B$ such that

$$M[G[u_1;\, t, x] - G[u_2;\, t, x]] \leqq A\, M(u_1 - u_2) + B\, |t_1 - t_2|$$

for all points $(t_1, t_2)$ and all functions $(u_1, u_2)$ of $H$.

(c) Corresponding to the function $u_0(x)$ in $H$ there exists an upper bound $D > 0$ for the quantity $MG[u_0;\, t, x]$ for $|t - t_0| \leqq k$.

THEOREM 7. *If* (a), (b), *and* (c) *are satisfied, then, for* $|t - t_0|$ *sufficiently small there exists uniquely, among functions of the class* $P$, *a function* $u(x,t)$ *which satisfies* (38) *and, also, the condition* $u(x, 0) = u_0(x)$.

* Paper II, Theorem IV.

Consider the system (40) corresponding to (38). As a consequence of ($a$) it is seen that, if $\underset{n=\infty}{\text{l.m.}}\, \zeta_n = \zeta$ and $\lim_{n=\infty} t_n = t$, then, if

$$(\zeta_n \equiv u_n,\ \zeta \equiv u)$$

$$0 = \lim_{n=\infty} \int_{(ab)} (G[u_n;\, t_n, x] - G[u;\, t, x])^2\, dx = \lim_{n=\infty} \sum_{i=1}^{\infty} [g_i(\zeta_n, t_n) - g_i(\zeta, t)]^2.$$

Therefore, each function $g_i(\zeta, t)$ possesses continuity, in the weak sense with respect to $\zeta$ and in the ordinary sense in $t$, simultaneously in its arguments. From ($b$) it follows that

$$\sum_{i=1}^{\infty} [g_i^2(\zeta_1, t_1) - g_i^2(\zeta_2, t_2)]^2 \leqq (AM(\zeta_1 - \zeta_2) + B\,|\,t_1 - t_2\,|)^2,$$

for all points $(\zeta_1, \zeta_2)$ in Hilbert space and for all $(t_1, t_2)$ on $|\,t - t_0\,| \leqq k$. As an obvious consequence of ($c$) it is seen that $D^2$ is an upper bound for

$$\sum_{i=1}^{\infty} g_i^2(\zeta_0, t). \qquad (\zeta_0 \equiv u_0;\ |\,t - t_0\,| \leqq k)$$

The consequences of (a), (b) and (c) just established constitute the assumptions under which we may state* that, for $|\,t - t_0\,|$ sufficiently small, (40) has a unique solution $\zeta = \zeta(t)$ $(\zeta(0) = \zeta_0)$, in Hilbert space, which satisfies (39) with $k_i = dz_i/dt$. From our previous discussion of the relationship between (38) and (40) it is seen that Theorem (7) has been completely established.

Various generalizations of the results of Parts I and II are immediately obvious. Instead of taking $H$ as the class of all integrable functions we could have restricted it to those functions $u$ satisfying $M(u - u_0) \leqq h$, where $u$ is a particular integrable function, and where $h$ is a given positive constant. Obvious changes would permit all proofs given to extend to this more general case. The variables $(s, x, t)$ used above to represent single variables could be thought of as being $m$-partite variables and the necessary modifications of the hypotheses in the work above are easily determined.

* Paper II, Theorem IV and Corollary (2) to Theorem IV.

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